THE SUPERMEMBRANE WITH CENTRAL CHARGES: (2+1)-D NCSYM, CONFINEMENT AND PHASE TRANSITION

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Abstract.
The spectrum of the Bosonic sector of the $D = 11$ supermembrane with central charges is shown to be discrete and with finite multiplicities, hence containing a mass gap. The result extends to the exact theory our previous proof of the similar property for the $SU(N)$ regularized model and strongly suggest discreteness of the spectrum for the complete Hamiltonian of the supermembrane with central charges. This theory is a quantum equivalent to a symplectic non-commutative Super Yang Mills in $2+1$ dimensions, where the space-like sector is a Riemann surface of positive genus. Along these lines, it is demonstrated how the theory exhibits confinement in the supermembrane with central charges phase and how the theory enters in the asymptotic-free phase through the spontaneous breaking of the center, which corresponds to the supermembrane without central charges.

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1. Introduction

A crucial step towards the understanding of the non-perturbative approach to Superstring Theory is the non-perturbative treatment of $D = 11$ supermembranes [1]. The quantization of the latter, when it is embedded on Minkowski space-time, was studied in [2, 3, 4, 5] in terms of a quantum mechanical maximally super-symmetric $SU(N)$ Yang-Mills matrix models. This was also considered on a different context in [6]. In the seminal work [2], it was shown that the spectrum of the $SU(N)$ regularized super-symmetric Hamiltonian is continuous, consisting of the interval $[0, \infty)$. Remarkably, the spectrum of the corresponding Bosonic Hamiltonian, equivalent to the dimensional reduction of $D = 10$ Super Yang-Mills to $0 + 1$ space-time, is discrete [7, 8]. However its configuration space contains singular configurations string-like spikes, which, along with super-symmetry, renders the spectrum continuous.

The validity of the $SU(N)$ regularization is justified by the fact that the structure constants of the area preserving diffeomorphisms, the gauge symmetry of the supermembrane in the light cone gauge (LCG), are equal to the large N limit of the $SU(N)$ structure constants. The characterization of the spectrum was performed on the $SU(N)$ regularized model, but we are not aware of any result concerning the large N limit of the spectrum.

The supermembrane embedded on a target space with a compact sector was analyzed in [9]. Although a $SU(N)$ regularization was not obtained, it was argued that the same qualitative features of the spectrum remain valid. The supermembrane theory was interpreted as an extended object theory. In this interpretation, the string-like spikes may connect different membranes without changing the energy of the system, in distinction in distinction to the standard case in String Theory. For a review see [10].

The $D = 11$ supermembrane with nontrivial central charges was introduced in [11]. The configuration space of this model is restricted by a topological condition. This restriction implies the existence of a non-trivial central charge on the SUSY algebra of the supermembrane. From a geometrical point of view, the topological condition determines a non-trivial $U(1)$ principal bundle over the worldvolume whose canonical connections, $U(1)$ monopoles, define minimal immersions into the compactified sector of the target space. These immersions describe the wrapping of the supermembrane on a calibrated sub-manifold of this target space [16].

The supermembrane with non-trivial central charges does not contain string-like spikes and it admits an $SU(N)$ regularization [12]. The Bosonic potential of this model increases towards infinity as we move away from zero in the configuration space, ensuring a compact resolvent for the Bosonic Hamiltonian [13]. The spectrum of the regularized Hamiltonian is discrete, with finite multiplicity [14] and its heat kernel can be defined rigorously by a process described in [15].

In the topological restriction on the configuration space, genus 2 and 3, $N = 1$ supermembranes with nontrivial central charges, correspond to the orthogonal intersection of a suitable number of genus 1 supermembranes with nontrivial central charges [16]. In the type IIA picture, the theory may be viewed as a bundle of $D2-D0$ bound state theories where $D0$ monopole charges are induced by non-constant fluxes on the $D2$ [17]. Extensions to $SU(N)$ interacting supermembranes may be considered as in [18].

In cases where a minimal immersion from the base manifold into the compact sector of the target space can be established (the former is a Riemann surface of genus g), the topological restriction can be solved, and the supermembrane with non-trivial central charges is equivalent (as a quantum field theory) to an $N = 1$ symplectic noncommutative Super Yang-Mills theory [19, 20]. This is the case, for instance, when the base manifold is a genus g Riemann surface and the compact sector of the target space is a flat torus T^{2g}. The symplectic structure is determined by the minimal immersion, and describes the curvature of $U(1)$ monopole connections. It is a non-constant θ parameter. Noncommutative Yang-Mills theories have been considered as toy models for gravity [21]. For a review see also [22]. The relation between supergravity and noncommutative Yang-Mills become natural in the context of supermembranes, since they are embedded on a target space which must be a solution of $D = 11$ Supergravity, moreover the supermultiplet of $D = 11$ Supergravity has been conjectured to be the ground state of the theory.

In the context of string theory, noncommutative SYM appear in a very natural way by wrapping D-branes [23]. A SYM theory on a noncommutative torus is naturally related to the compactification of a matrix theory on a dual torus with a constant C_3 field, see for example, [24],[26],[25],[27]. NCYM theory in a flat space with a rational noncommutative parameter is related to ordinary Yang-Mills theories with magnetic flux through Morita equivalence, [28]. By comparing ordinary Yang-Mills theories (YM) and noncommutative ones (NCYM), it was found in [29] that both theories share the same degrees of freedom in the IR limit although in the UV one, those degrees are redistributed differently in both theories. The hierarchy between noncommutative and commutative theories naively have being thought to correspond respectively to the high energy limit and lowering the scale we recover the commutative space. However it was argued by [30, 27] that it should be the noncommutative Yang Mills the one more appropriate to describe the IR limit of the theory, while the commutative YM the UV.

BFSS conjecture takes the $D0$ action as the fundamental action [31]. It coincides with the $D = 11$ supermembrane matrix formulation in the light cone gauge. This point of view has allowed to extend matrix models, from an effective point of view, to interesting compactified manifolds. A good example is [32] BMN model, in which additional mass terms to BFSS conjecture that respect pp-wave supersymmetry were added. These extra terms are Chern-Simons and mass terms, and due to its presence, stable vacuum solutions were found which were interpreted as spherical branes.

In the construction of the supermembrane with central charges it has been relevant not only the structure of the compact sector of the target space, but also the topology of the base manifold as well as the minimal immersion realizing the wrapping of the supermembrane on a calibrated submanifold of the target. The geometrical structure is lacking in the matrix model approach, and will be important when we analyze the large N limit of the regularized models, in particular when we determine the geometrical structure of the configuration space.

In this paper we will prove that the bosonic Hamiltonian of the supermembrane with non-trivial central charges has discrete spectrum with finite multiplicity. Moreover its resolvent is compact. We will argue that the spectrum of the supersymmetric Hamiltonian has qualitatively the same properties. Consequently the NCSYM in $2+1$ share those properties. The large N limit of the eigenvalues of the semi-classical regularized Hamiltonian converge one to one to the eigenvalues of the semi-classical exact Hamiltonian of the supermembrane with central charges. The spectrum of the Hamiltonian exhibits a mass gap and the scalar fields acquire a mass induced by the center $Z(2)$ of the symplectic group in the IR phase, equivalently the $Z(N) \times Z(N)$ in the regularized model. However rising the energy we will explicitly show how the center breaks spontaneously and a transition phase happens ending on a screening one. This corresponds to have a $N = 4$ compactified supermembrane without central charges. In the bosonic phase it still shows a deformed mass gap (as it should happen, see [21]) and it is exclusively induced by the moment of inertia of the membranes [8], but in the supersymmetric case the picture is even clearer since the spectrum in the screening phase is purely continuous. The NCSYM theory in $2+1$ shares similar confinement properties as susy QCD.

YM theories with boundary modelling QCD were extensively studied long time ago based on a model originally called bag model [41]. SYM theories behave very differently in the low or high energies since they are in the confined or screening phase. As explained in [33], the confined phase corresponds for the bosonic theory to the phase of low temperatures at which vector-like gluons form singlet bound states of color which are called glueballs. They appear many times in the adjoint representation. The low temperature regime is characterized by the dynamics of gluons. At high temperatures the gluons are forming no more bound states but form a plasma that constitute the screening phase. If fermions are introduced in the theory (SYM) they feel a binding force against being separated at low energies and are free in the high energy regime [35]. These two regimes in general are thought to be separated by a phase transition that happens when a global symmetry of the theory breaks and it is related with the spontaneous breaking of the center as first pointed by [35, 36]. In this works it was pointed that the nature of this symmetry was conjectured to be topological due to magnetic monopoles or instantons. In several papers the role of the center was studied [37, 33, 38]. In [39] he pointed out that the instantons gas picture is only appropriate in those case in which the topological charge is discrete, otherwise the correct one would be a monopole picture. He argued that this last case should be the one in which the confinement should appear. This is in fact the case of the supermembrane with central charges. A previous attempt trying to connect membrane theory with YM theories was done by [40] in a different context. In [42] they relate the critical behavior of a gauge theory in the de-confined phase with the behavior of a scalar which has a symmetry induced by the center of the group. The transition phase happens when the topological defect is metastable and decay through quantum processes.

The paper is organized as follows. In Section 2, we obtain the semi-classical regime of the supermembrane with central charges. In Section 3, we find the operator bounds on the exact bosonic hamiltonian, in section 4 we obtain the semi-classical approximation of the regularized model. In Section 5 we find the large N limit of the semi-classical bosonic Hamiltonian. In section 6 we study the confinement properties of the theory in terms of the center of the group at the exact and regularized level and the transition phase to de-confinement and give an interpretation in terms of supermembranes. In section 7 we discuss our results and conclude.

2. The supermembranes with central charges and its semi-classical regime

In this section we analyze the semi-classical approximation of the exact action of the supermembrane with central charges. Our main concern will be the semi-classical quantization of the eleven dimensional supermembrane compactified on a torus.

Let the $D = 11$ supermembrane be defined in terms of a base manifold, a $g = 1$ Riemann surface Σ, and a target space $M_9 \times S^1 \times S^1$. Consider its formulation in the light cone gauge where the directions X^+, X^-, P_+ and P_- have been removed in the standard manner [4]. The canonically reduced Hamiltonian has the expression

$$\int_\Sigma \sqrt{W}\left(\frac{1}{2}\left(\frac{P_M}{\sqrt{W}}\right)^2 + \frac{1}{4}\{X^M, X^N\}^2 + \text{ Fermionic terms}\right) \tag{1}$$

subject to the condition

$$\oint_{\mathcal{C}} \frac{P_M}{\sqrt{W}} \mathrm{d}X^M = 0. \tag{2}$$

Here and below $M, N = 1, \ldots, 9$. The integral on the left side of (2) is the generator of an area preserving diffeomorphism of Σ for $\mathcal{C}$ any given closed path. This constraint may be decomposed into a local condition

$$\{P_M, X^M\} \equiv \epsilon^{ab}\partial_a\left(\frac{P_M}{\sqrt{W}}\right)\partial_b X^M = 0 \tag{3}$$

which generates area preserving diffeomorphisms connected to the identity, coupled with the constraint

$$\oint_{C_i} \frac{P_M}{\sqrt{W}} \mathrm{d}X^M = 0, \qquad i = 1, 2, \tag{4}$$

where C_1 and C_2 form of a basis of homology on Σ which generates area preserving diffeomorphisms disconnected from the identity.

The scalar density $\sqrt{W}$ is present in expression (4) as a consequence of the gauge fixing procedure and it is preserved by the above diffeomorphisms. Let us now impose some topological restrictions on the configuration space which completely characterize the $D = 11$ supermembrane with non-trivial central charge generated by the wrapping on the compact sector of the target space. All maps from the base space Σ, must satisfy

$$\begin{aligned} &\oint_{C_i} \mathrm{d}X^r = 2\pi S_i^r R^r, && r = 1,2, \\ &\oint_{C_i} \mathrm{d}X^m = 0 && m = 3,\ldots,9 \end{aligned} \tag{5}$$

for $i = 1,2$ and

$$\int_\Sigma \mathrm{d}X^r \wedge \mathrm{d}X^s = \epsilon^{rs}(2\pi^2 R_1 R_2)n, \tag{6}$$

where $n = \det S_i^r$ is fixed, each entry S_i^r is integer, and R_1 and R_2 denote the radii of the target component $S^1 \times S^1$. Note that (5) describe maps from Σ to $S^1 \times S^1$ with $\mathrm{d}X^m$ a non-trivial closed one-form. The only restriction upon these maps is the assumption that n is fixed. The term on the left side of (6) describes the central charge of the supersymmetric algebra. As we shall see next, the factor $R_1 R_2 (2\pi)^2$ is the area of Σ in the induced metric.

The general map satisfying (5-6) can be constructed explicitly. Any closed one-form $\mathrm{d}X^r$ decomposes into the sum of a harmonic and an exact form,

$$\mathrm{d}X^r = L_s^r \mathrm{d}\hat{X}^s + \delta_s^r \mathrm{d}A_s \qquad s, r = 1,2 \tag{7}$$

where L_s^r are real numbers and $\mathrm{d}\hat{X}$ is a canonical basis of harmonic one-forms over Σ. The term $\mathrm{d}\hat{X}^s$, $s = 1,2$, is found by considering the (unique) holomorphic one-form ω, normalized with respect to the elements of the homology basis C_i, defined by:

$$\oint_{C_1} w = 1, \qquad \oint_{C_2} w = \Pi, \tag{8}$$

where Π is the period of ω in the basis given by C_i. By construction, the imaginary part of Π is positive. Let [16]

$$\omega = \mathrm{d}\hat{X}^1 + i\mathrm{d}\hat{X}^2 \tag{9}$$

and define

$$\mathrm{d}\hat{X}^r = (M^{-1}\mathrm{d}\tilde{X})^r, \tag{10}$$

where the constant matrix M is given by

$$M = \begin{pmatrix} 1 & \operatorname{Re}\Pi \\ 0 & \operatorname{Im}\Pi \end{pmatrix}. \tag{11}$$

Then [16]

$$\oint_{C_i} \mathrm{d}\hat{X}^r = \delta_i^r \tag{12}$$

and

$$\int_\Sigma \mathrm{d}\hat{X}^r \wedge \mathrm{d}\hat{X}^s = \epsilon^{rs}. \tag{13}$$

If (5) is to be satisfied, necessarily

$$L^r_s = 2\pi R^r S^r_s. \tag{14}$$

Condition (6) implies

$$S^r_s S^t_u \epsilon^{su} = n\epsilon^{rt}. \tag{15}$$

Define the scalar density $\sqrt{W}$ by

$$\epsilon^{ab}\partial_a \hat{X}^r \partial_b \hat{X}^s \epsilon_{rs} = \sqrt{W}, \tag{16}$$

where $\partial_a \equiv \partial/\partial\sigma^a$, $a = 1, 2$, σ^a are local coordinates on Σ. Then

$$\epsilon_{rs}\ \mathrm{d}\hat{X}^r \wedge \mathrm{d}\hat{X}^s = \sqrt{W}\ \mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2. \tag{17}$$

A change in the canonical basis of homology over Σ, implies varying the corresponding harmonic one-form $\mathrm{d}\hat{X}^r \rightsquigarrow T^r_s \mathrm{d}\hat{X}^s$, where $T \in SL(2, Z)$, that is

$$T^r_s T^t_u \epsilon^{su} = \epsilon^{rt}, \tag{18}$$

T^r_s integers. The density $\sqrt{W}$ remains invariant under these transformations consequently they are area-preserving diffeomorphisms disconnected from the identity. The theory is then invariant under $SL(2, Z)$. The degrees of freedom are expressed in terms of A_r and the discrete set of integers described by the harmonic one-forms. We can always fix these gauge transformations by

$$S^r_i = l^r \delta^r_i, \qquad l^1 l^2 = n. \tag{19}$$

Therefore

$$\mathrm{d}X^r = 2\pi R^r l^r \mathrm{d}\hat{X}^r + \delta^r_s \mathrm{d}A_s. \tag{20}$$

After the gauge fixing there is a residual invariance $\mathbb{Z}(2)$.

The complete expression for the Hamiltonian of the $D = 11$ supermembrane subject to the topological conditions (5) and (6) turns out to be [14], [19], [12],

(21)

$$H = \int \sqrt{W}\mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2 \left[\frac{1}{2}\left(\frac{P_m}{\sqrt{W}}\right)^2 + \frac{1}{2}\left(\frac{\Pi^r}{\sqrt{W}}\right)^2 + \frac{1}{4}\{X^m, X^n\}^2 + \frac{1}{2}(\mathcal{D}_r X^m)^2 + \frac{1}{4}(\mathcal{F}_{rs})^2 + \Lambda(\{P_m, X^m\} + \mathcal{D}_r \Pi^r)\right] + \text{ Fermionic term.}$$

where [19], [12]

$$\begin{aligned}\mathcal{D}_r X^m &= 2\pi R^r l^r \frac{\epsilon^{ab}}{\sqrt{W}} \partial_a \hat{X}^r \partial_b X^m,\\ \mathcal{D}_r \Pi^r &= 2\pi R^r l^r \frac{\epsilon^{ab}}{\sqrt{W}} \partial_a \hat{X}^r \partial_b \left(\frac{\Pi^r}{\sqrt{W}}\right) + [A_r, \Pi^r],\\ \mathcal{F}_{rs} &= \mathcal{D}_r A_s - \mathcal{D}_s A_r + [A_r, A_s].\end{aligned} \tag{22}$$

The associated mass operator is $mass^2 = Z^2 + H$, where Z is the central charge $\frac{1}{8}(nR_1R_2)^2$.

The semi-classical approximation of the theory is obtained by only considering the quadratic terms in the above expression for the Hamiltonian. Let

$$\begin{aligned} H_{\mathrm{sc}} = \int \sqrt{W} \mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2 \Bigg[&\frac{1}{2}\left(\frac{P_m}{\sqrt{W}}\right)^2 + \frac{1}{2}\left(\frac{\Pi^r}{\sqrt{W}}\right)^2 + \\ &\frac{1}{2}(\mathcal{D}_r X^m)^2 + \frac{1}{4}(\widehat{\mathcal{F}}_{rs})^2 + \Lambda \mathcal{D}_r \Pi^r \Bigg] + \text{ Fermionic terms,}\end{aligned} \tag{23}$$

where in the semi-classical approximation

$$\widehat{\mathcal{F}}_{rs} = D_r A_s - D_s A_r. \tag{24}$$

The general solution to the constraint $\mathcal{D}_r \Pi^r = 0$ is

$$\Pi^r = \epsilon^{rs} 2\pi R^s l^s \epsilon^{ab} \partial_a \hat{X}^s \partial_b \left(\frac{\Pi}{\sqrt{W}}\right) \tag{25}$$

where Π is a scalar density.

The kinetic term $\Pi^r \dot{A}_r$ may be rewritten, after integration by parts, as $p\dot{q}$ where $p = \sqrt{W}\frac{1}{2}\epsilon^{rs}\mathcal{F}_{rs}$ and $q = \frac{\Pi}{\sqrt{W}}$. This yields

$$H = \int \mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2 \sqrt{W} \left[\frac{1}{2}\left(\frac{\Pi^r}{\sqrt{W}}\right)^2 + \frac{1}{4}(\widehat{\mathcal{F}}_{rs})^2 + \Lambda \mathcal{D}_r \Pi^r\right] = \tag{26}$$

$$= \int \mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2 \sqrt{W} \left[\frac{1}{2}\left(\frac{p}{\sqrt{W}}\right)^2 + \frac{1}{2}(\mathcal{D}_r q)^2\right] \tag{27}$$

which coincides with the contribution to the Hamiltonian of the transverse modes X^m, $m = 3, \ldots, 9$.

The above shows that, from a gauge independent point of view, the complete Bosonic Hamiltonian in the semi-classical approximation is

$$H_{\mathrm{sc}}^{B} = \int \mathrm{d}\sigma^1 \wedge \mathrm{d}\sigma^2 \sqrt{W} \left[\frac{1}{2} \left(\frac{P^M}{\sqrt{W}} \right)^2 + \frac{1}{2} (\mathcal{D}_r X^M)^2 \right] \tag{28}$$

where $M = 1, \ldots, 8$. If we now express X^M and $P^M/\sqrt{W}$ in terms of a complete orthonormal basis of scalar symmetries over over Σ, we obtain

$$X^M = X_A^M(\tau) \exp[2\pi i(a_r \hat{X}^r)](\sigma), \tag{29}$$

$$\frac{P^M}{\sqrt{W}} = \rho_A^M(\tau) \exp[2\pi i(a_r \hat{X}^r)](\sigma), \tag{30}$$

where $A = (a_1, a_2)$. Thus, the Bosonic contribution in the semi-classical Hamiltonian takes the form

$$H_{\mathrm{sc}}^{\mathrm{B}} = [(\rho_A^M)^2 + \omega_A^2 (X_A^m)^2]. \tag{31}$$

The spectrum of $H_{\mathrm{sc}}^{\mathrm{B}}$ is then characterized in the following fashion. For any finite subset Ω of $\mathbb{N} \times \mathbb{N}$ there is an eigenvalue

$$\begin{gathered} \lambda_\Omega = \sum_{A \in \Omega} \omega_A, \\ \omega_A = \pi^2 \sqrt{(R^1 l^1 a_2)^2 + (R^2 l^2 a_1)^2}. \end{gathered} \tag{32}$$

This expression coincides with the particular case considered in [47]. By virtue of (32), for any given energy level E, there only exists a finite number of eigenvalues of $H_{\mathrm{sc}}^{\mathrm{B}}$ below E.

This is the expression of the eigenvalues when the zero point energy has been eliminated. It is authomatically cancelled when the semi-classical supersymmetric hamiltonian is considered. This property was first proven in [47] and it is exactly the same for the semiclassical supermembrane with central charges.

3. Operator bounds on the exact bosonic Hamiltonian

According to the results reported in [13], the bosonic regularized Hamiltonian of the $D = 11$ supermembrane with central charge, H_N^B, relates to its semi-classical approximation, $H_{\mathrm{sc},N}^B$, by means of the following operator inequality:

$$H_N^B \geq C_N H_{\mathrm{sc},N}^B. \tag{33}$$

Here N denotes the size of the truncation in the Fourier basis of Σ and C_N is a positive constant. A seemingly crucial step in the proof of (33) found in [13], relies heavily on the compactness of the unit ball of the configuration space which happens to be finite dimensional. In this section we show that the same operator relation holds true for the exact bosonic Hamiltonians, see Theorem 1. The main source of difficulties in the proof of Theorem 1 lies in the fact that now the unit ball of the configuration space does not posses the property of being compact. We overcome these difficulties by carrying out a detailed analysis of each term involved in the expansion of the potential term of H^B.

Before proceeding further, we should remark that it is believed that the compactness of the configuration space for Yang Mills theories implies a mass gap in the spectrum. We are not aware of any complete proof of such assertion.

3.1. **The configuration space and the gauge fixing condition.** We define the configuration space for the supermembrane with central charges in the following fashion.

Since constant functions are harmonic, the decomposition into harmonic and exact one-forms discussed in Section 2 ensures that the constant modes of the fields X^m and A_r are to be included in the harmonic sector. Let $\mathcal{H}^1$ denote the Hilbert space obtained by completing $C^1(\Sigma)$ modulo locally constant functions, with respect to the norm

$$\|u\|^2 = \int \mathrm{d}^2\sigma \sqrt{w} g^{ab} \partial_a u \partial_b \overline{u}, \tag{34}$$

where g^{ab} is the inverse of the metric $g_{ab} = \partial_a \widehat{X}^r \partial_b \widehat{X}^r$ induced over Σ by the minimal immersion $\widehat{X}_r$. Below we use the following convention: for a field u,

$$\langle u \rangle = \int_\Sigma \mathrm{d}^2\sigma \sqrt{w}\, u.$$

Notice that

$$\begin{aligned} D_r u D_r \overline{u} &= \frac{\epsilon^{ab}}{\sqrt{w}} \partial_a \widehat{X}^r \partial_c u \frac{\epsilon^{bd}}{\sqrt{w}} \partial_b \widehat{X}^r \partial_d \overline{u} \\ &= g^{cd} \partial_c u \partial_d \overline{u} \end{aligned} \tag{35}$$

and

$$\{u, w\} \equiv \frac{\epsilon^{ab}}{\sqrt{w}} \partial_a u \partial_b w = \epsilon^{rs} D_r u D_s w, \tag{36}$$

so that

$$\|u\|^2 = \langle D_r u D_r \overline{u} \rangle. \tag{37}$$

Following the standard notation, for $p = 2, 4$, $L^p \equiv L^p(\sigma)$ denotes the Banach space of all fields u, such that

$$\|u\|_p = \langle u^p \rangle^{1/p} < \infty.$$

Let

$$\|u\|_{4,2} = (\|D_r u\|^4 + \|D_r D_s u\|^4)^{1/4}. \tag{38}$$

Below and elsewhere the fields X^m, A_r will be assume to lie on the *configuration space* $\mathcal{H}^{4,2}$ of functions $u \in \mathcal{H}^1$ such that $\|u\|_{4,2} < \infty$. Note that the left hand side of (38) is a well defined norm in $\mathcal{H}^{4,2}$, the latter is a linear space, but we do not make any assumption about completeness.

The potential, V, of the bosonic sector of the supermembrane with central charges is well defined in $\mathcal{H}^{4,2}$ as

$$V = \langle \mathcal{D}_r X^m \mathcal{D}_r X^m + \frac{1}{4}\mathcal{F}_{rs}\mathcal{F}_{rs}\rangle. \tag{39}$$

The introduction of the constrained space $\mathcal{H}^{4,2}$ is justified by the fact that homogeneous terms of order 4 are present on the right hand side of (39). Indeed, V is not well defined in $\mathcal{H}^1$. However, below we will appeal extensively to the Euclidean properties of the latter Hilbert space.

The following gauge fixing conditions,

$$\begin{aligned} &D_1 A_1 = 0 \\ &D_1 A_2 = 0 \Rightarrow A_2 = 0, \end{aligned} \tag{40}$$

are equivalent to those considered in [12, 13, 14], which are obtained by expressing the fields in terms of an orthonormal basis over Σ. Integration by parts yields

$$\langle D_2 A_1 D_1 A_2 \rangle = 0. \tag{41}$$

Similarly we also have

$$\langle D_2 A_1 \{A_1, A_2\}\rangle = 0. \tag{42}$$

Note that

$$\langle (D_1 A_2)^2 \rangle = 0, \tag{43}$$

implies $A_2 = 0$.

3.2. **The uniform quadratic bound for the Bosonic potential.** Let ρ^2 be the potential term of H^B_{sc}, so that

$$\rho^2 = \langle D_r X^m D_r X^m + (D_1 A_2)^2 + (D_2 A_1)^2 \rangle. \tag{44}$$

We may rewrite

$$V = \rho^2 + 2\mathbf{B} + \mathbf{A}^2 \tag{45}$$

where

$$\mathbf{B} = \langle D_r X^m \{A_r, X^m\} + D_1 A_2 \{A_1, A_2\}\rangle, \tag{46}$$

$$\mathbf{A} = \langle \{A_1, X^m\}^2 + \{A_2, X^m\}^2 + \{A_1, A_2\}^2 + \{X^m, X^n\}^2\rangle. \tag{47}$$

Theorem 1. *There exists a constant* $0 < C \leq 1$*, such that*

$$V \geq C\rho^2, \qquad \forall X^m,\ A_r \in \mathcal{H}^2. \tag{48}$$

We devote the remaining parts of this section to show the validity of Theorem 1.

Note that

$$\begin{aligned} V &= \rho^2 \left(1 + \frac{2\mathbf{B}}{\rho^2} + \frac{\mathbf{A}^2}{\rho^2}\right) \\ &= \rho^2 (1 + 2bR + a^2 R^2), \end{aligned}$$

where

$$R = \|(X^m, A_r)\|_{4,2}, \tag{49}$$

$$a^2 = \frac{\mathbf{A}^2}{R^2 \rho^2} \qquad \text{and} \qquad b = \frac{\mathbf{B}}{\rho^2 R}. \tag{50}$$

Since both terms a^2 and b are homogeneous in X^m and A_r, they are constant in R. Without loss of generality we assume that a^2 and b are evaluated at fields X^m, A_r normalized by the condition $R = 1$.

Let

$$P(R) = 1 + 2bR + a^2 R^2$$

be the real polynomial whose variable is $R \geq 0$. Demonstrating the existence of a constant $C > 0$ satisfying (48) is equivalent to showing that

$$\inf_{\|(X^m, A_r)\|_{4,2} = 1} \left[\inf_{R \geq 0} P(R) \right] \geq C. \tag{51}$$

Notice that $\mathbf{B}$ is the inner product in $\mathcal{H}_1$ of the field $(D_1 X^m, D_2 X^m, D_1 A_2, 0)$ times $(\{A_1, X^m\}, \{A_2, X^m\}, \{A_1, A_2\}, \{X^m, X^n\})$ while $\mathbf{A}^2$ is the norm of the latter. Thus $A^2 = 0$ yields $B = 0$, so the condition $a^2 = 0$ implies $P(R) = 1$. Hence

$$\inf_{\|(X^m, A_r)\|_{4,2} = 1} \left[\inf_{R \geq 0} P(R) \right] = \min \left[1, \inf_{a \neq 0} \left(1 - \frac{b^2}{a^2} \right) \right]$$

in (51).

The validity of the following lemma will immediately ensure (51), hence Theorem 1.

Lemma 2. *Let a and b be the quantities defined by* (50). *Then*

$$\inf_{a\neq 0}\left(1-\frac{b^2}{a^2}\right)>0.$$

Proof. We proceed by contradiction. Suppose that

$$\inf_{a\neq 0}\left(1-\frac{b^2}{a^2}\right)=0.$$

Then we can find a sequence $(X^m)_j$, $(A_r)_j$ in the configuration space, such that $\|((X^m)_j,(A_r)_j)\|_{4,2}=1$ and

$$\frac{b_j^2}{a_j^2}\to 1 \tag{52}$$

as $j\to\infty$, in the obvious notation.

Now

$$\frac{b^2}{a^2}=\frac{\langle D_rX^m\{A_r,X^m\}+D_1A_2\{A_1,A_2\}+(D_2A_1)0+0\{X^m,X^n\}\rangle^2}{\rho^2\mathbf{A}^2}.$$

So, for each $j=1,2,\ldots$, the left hand side of (52) is the inner product of two vectors of norm equal to 1 in $\mathcal{H}^1$. By virtue of the Cauchy Schwarz inequality, these two vectors should become increasingly parallel as as $j\to\infty$. Since the quantities a^2 and b^2 remain constant if we multiply the field (X^m,A_r) by a constant, without loss of generality we can chose our sequence, such that

$$\left\langle\left(\frac{D_r(X^m)_j}{\rho_j}-\frac{\{(A_r)_j,(X^m)_j\}}{\mathbf{A}_j}\right)^2\right\rangle\to 0, \tag{53}$$

$$\left\langle\left(\frac{D_1(A_2)_j}{\rho_j}-\frac{\{(A_1)_j,(A_2)_j\}}{\mathbf{A}_j}\right)^2\right\rangle\to 0, \tag{54}$$

$$\left\langle\left(\frac{D_2(A_1)_j}{\rho_j}\right)^2\right\rangle\to 0\quad\text{and}\quad\left\langle\left(\frac{\{(X^m)_j,(X^n)_j\}}{\mathbf{A}_j}\right)^2\right\rangle\to 0. \tag{55}$$

When $r=1$, the left side of (53), is

$$\left\langle\left(\frac{D_1(X^m)_j}{\rho_j}-\frac{D_2(A_1)_jD_1(X^m)_j}{\mathbf{A}_j}\right)^2\right\rangle\geq$$
$$\left\langle\left(\frac{D_1(X^m)_j}{\rho_j}\right)^2\right\rangle-2\left|\left\langle\frac{D_1(X^m)_j}{\rho_j}\frac{D_2(A_1)_jD_1(X^m)_j}{\mathbf{A}_j}\right\rangle\right|.$$

By virtue of the Cauchy Schwarz inequality,

$$\left|\left\langle \frac{D_1(X^m)_j}{\rho_j} \ \frac{D_2(A_1)_j D_1(X^m)_j}{\mathbf{A}_j} \right\rangle\right| = \left|\left\langle (D_1(X^m)_j)^2 \frac{D_2(A_1)_j}{\rho_j \mathbf{A}_j} \right\rangle\right|$$
$$\leq \langle (D_1(X^m)_j)^4 \rangle^{1/2} \left\langle \left(\frac{D_2(A_1)_j}{\rho_j}\right)^2 \right\rangle^{1/2} \frac{1}{\mathbf{A}_j}$$
$$\leq \left\langle \left(\frac{D_2(A_1)_j}{\rho_j}\right)^2 \right\rangle^{1/2} \frac{1}{\mathbf{A}_j}.$$

Furthermore, analogous results hold for the right hand sides of (53) with $r = 2$ and (54). Hence, if $\mathbf{A}_j \not\to 0$, the above, along with (53) and (54), imply

$$\frac{\langle (D_r(X^m)_j)^2 + (D_1(A_2)_j)^2 + (D_2(A_1)_j)^2 \rangle}{\rho_j^2} \to 0$$

which is impossible.

It is only left showing that the case $\mathbf{A}_j \to 0$ also produces a contradiction. We proceed as follows.

Let Δ denote the Laplacian operator acting on $L^2(\Sigma)$. Integration by parts show that

$$\langle D_r(D_1(X^m)_j)^2 D_r(D_1(X^m)_j)^2 \rangle = \langle [(-\Delta)^{1/2}(D_1(X^m)_j)^2]^2 \rangle.$$

Then, $(W^m)_j = (-\Delta)^{1/2}(D_1(X^m)_j)^2 \in L^2(\Sigma)$. Since $\|(X^m)_j\|_{4,2} \leq 1$, $\|(W^m)_j\|_2 \leq 1$. Now $(-\Delta)^{-1/2}$ is a compact operator and $(D_1(X^m)_j)^2 = (-\Delta)^{-1/2}(W^m)_j$. Thus $(D_1(X^m)_j)^2$ has a subsequence which is convergent in $\|\cdot\|_2$ to an accumulation point, say $Y_1^m \in L^2(\Sigma)$.

Similarly $(D_2(X^m)_j)^2$, $(D_1(A_2)_j)^2$ and $(D_2(A_1)_j)^2$ have corresponding subsequences, convergent in $\|\cdot\|_2$ to accumulation points, Y_2^m, Z_1 and Z_2 in $L^2(\Sigma)$. Furthermore Y_r^m and Z_r lie on $L^4(\Sigma)$, so that we can evaluate P at $((Y_r^m)^{1/2}, (Z_r)^{1/2})$. As $\mathbf{A}^2 = 0$ when we evaluate at these limit fields, in fact P achieves the constant value 1. But, since b and a^2 are continuous in (D_rX^m, D_1A_2, D_2A_1) for the norm $\|\cdot\|_2$, this contradicts the condition $b_j^2/a_j^2 \to 1$. This completes the proof of the lemma.

One important point is to define the Laplacian on the non-compact infinite dimensional configuration space we have introduced. We may proceed as follows. The hamiltonian is expressed as

$$H^B = [V_{quartic} + V_{cubic} + (1-C)V_{quadratic}] + [-\Delta + CV_{quadratic}]$$

where the first bracket acts multiplicatively on the Hilbert space of states while the operator on the second bracket may be expressed in terms of creation and annihilation operators in the usual way. The inequality we have proven is

$$H^B \geq -\Delta + CV_{quadratic}$$

we may now extract in a consistent way the infinite zero point energy from the same operator on both sides of the inequality. The zero point energy will be automatically cancelled when we considered the supersymmetric theory. We have proven ([14],[15]) on the regularized model that the fermionic contribution does not change the qualitatively properties of the bosonic hamiltonian. We expect to extend that arguments to the exact supermembrane with central charges. We will report on this elsewhere. The operator inequality implies that the spectrum of the exact theory is discrete with finite multiplicity. Moreover its resolvent is compact. The same inequality was proven for the regularized bosonic model. In order to relate both approaches we consider in the next section the regularized semiclassical model and discussed its large N limit.

4. The semi-classical approximation of the regularized model

Our first step consists in extracting quadratic terms from the complete regularized Hamiltonian of the supermembrane with central charges. The semi-classical Hamiltonian in the regularized model is,

$$\begin{aligned} H_{\mathrm{sc},N} = \mathrm{Tr}\,(&\frac{1}{2N^3}((P^m)^2 + (\Pi_r)^2) + \frac{n^2}{8\pi^2 N^3}(\frac{i}{N}[T_{V_r}, X^m]T_{-V_r})^2 \\ &+ \frac{n^2}{16\pi^2 N^3}(\frac{i}{N})^2([T_{V_r}, A_r]T_{-V_s} - [T_{V_s}, A_r]T_{-V_r})^2) \\ &- \frac{i}{N}(\frac{in}{4\pi N^3})\overline{\Psi}\gamma_-\gamma_r[T_{V_r}, \Psi]T_{-V_r}). \end{aligned} \tag{56}$$

We use the $SU(N)$ matrices $T_A = N\omega^{\frac{1}{2}a_1 a_2}P^{a_1}Q^{a_2}$, and $T_0 = N\mathcal{I}$ and $A = a_1, a_2$. P, Q are the Heisenberg matrices satisfying the Weyl condition $PQ = \omega QP$ where $\omega = e^{\frac{2\pi i}{N}}$. The generators of the algebra of $SU(N$ may be expressed in terms of T_A. The fields are expanded on this basis as

$$X^m = X^{mA}T_A \tag{57}$$

satisfying

$$\begin{aligned} [T_{V_r}, T_A] &= f_{V_r A}^{V_r + A} T_{V_r + A} \\ T_A T_B &= N\omega^{\frac{1}{2}A\wedge B} T_{A+B} \end{aligned} \tag{58}$$

where $f_{AB}^{C} = 2iN\sin(\frac{A\wedge B\pi}{N})\delta(A+B-C)$ are the standard structure constants for $SU(N)$ regularizing a two-torus, $A\wedge B \equiv a_1b_2 - a_2b_1$ and the vectors $V_{r=1} = (1,0), V_{r=2} = (0,1)$.
We analyze first the bosonic contribution to the above Hamiltonian. The bosonic contribution consists of two pieces coming from the transverse field sector and from the induced gauge fields on the world-volume of the membrane. Let us consider in first place the transverse field contribution,

$$\mathrm{Tr}\,(\frac{n^2}{8\pi^2N^3}(\frac{i}{N}[T_{V_r}, X^m]T_{-V_r})^2), \tag{59}$$

by performing straightforward calculations we find it is equivalent to

$$H^B_{\mathrm{sc}_1,N} = \sum_A |X^{mA}|^2 w_A^2 \tag{60}$$

where

$$w_A = \frac{nN}{\sqrt{2}\pi}\sqrt{\sin^2(\frac{a_1\pi}{N}) + \sin^2(\frac{a_2\pi}{N})}. \tag{61}$$

The second contribution, corresponds to the gauge fields defined on the world volume of the membrane as a result of the central charges induced by the winding. This contribution is

$$H^B_{\mathrm{sc}_2,N} = \frac{n^2}{8\pi^2N^5}\mathrm{Tr}\,([T_{V_1}, A_2]T_{-V_1} - [T_{V_2}, A_1]T_{-V_2})^2) \tag{62}$$

(63)

We have the freedom to fix the remaining gauge fields by imposing the following constraint

$$A_1^{(m,n)} = 0 \quad n \neq 0; \quad A_2^{(p,q)} = 0 \quad q = 0 \tag{64}$$

Then, for $r = 1$ the commutator is equal to

$$[T_{V_1}, A_2]T_{-V_1} = N\sum_B f_{V_1B}^{V_1+B}\omega^{+\frac{1}{2}b_2}T_B \tag{65}$$

Performing an analogous calculation for $r = 2$, we find that,using that the two terms in eqn. (62) are orthogonal,

$$H^B_{\mathrm{sc}_2,N} = \sum_B A_1^B\overline{A_1^B}w_{1B}^2 + A_2^B\overline{A_2^B}w_{2B}^2 \tag{66}$$

where

$$
\begin{aligned}
w_{1B} &= \frac{nN}{\sqrt{2\pi}} \sin(\frac{b_1\pi}{N}) \\
w_{2B} &= \frac{nN}{\sqrt{2\pi}} \sin(\frac{b_2\pi}{N})
\end{aligned} \tag{67}
$$

which are equal to the contributions of the X^m modes.

5. The Large N Limit of the Semi-classical Bosonic Hamiltonian

By virtue of the discussion carried out in Section 3.1, we see that the Bosonic regularized semi-classical Hamiltonian realizes as the quantum mechanical harmonic oscillator acting on the Hilbert space $L^2(\mathbb{R}^\Lambda, \mathbb{C})$,

$$
H^{\mathrm{B}}_{\mathrm{sc},N} = -\nabla^2_Y + (\omega_{A,N})^2 (Y^A)^2 + c_N. \tag{68}
$$

Here we agree in using the following convention: N is a large parameter representing the number of $D0$ branes in the regularization process; $Y = (Y^A) \in \mathbb{R}^\Lambda$ lies in the space coordinates; and there is as many as $\Lambda = N^2 - 1$ indexes A at each stage N. Here $A = (m, n)$. The constant c_N is a shift in the position of the ground state energy. We choose this constant at each stage N so that the ground energy of $H^{\mathrm{B}}_{\mathrm{sc},N}$ is exactly zero.

This characterization of $H^{\mathrm{B}}_{\mathrm{sc},N}$ as an elliptic partial differential operator is convenient when one aims at describing in rigorous manner properties of the spectrum [14] and heat kernel [15] of the Hamiltonian H, once the regularization process has been carried out. We now consider this representation in order to study the large N limit of $H^{\mathrm{B}}_{\mathrm{sc},N}$ and its connection with $H^{\mathrm{B}}_{\mathrm{sc}}$. All limiting process below refer to taking $N \to \infty$.

Lemma 3. *Each eigenvalue of $H^B_{\mathrm{sc},N}$ converges to a corresponding eigenvalue of H^B_{sc} as $N \to \infty$.*

Proof. Firstly note that there is a one to one correspondence between individual exited state eigenvalues of $H^B_{\mathrm{sc,N}}$ and H^B_{sc}, and finite subsets of $\mathbb{N} \times \mathbb{N}$. When $N < \infty$, the variables in (68) can be separated, so the eigenfunctions of $H^{\mathrm{B}}_{\mathrm{sc},N}$ are given explicitly in terms of creation operators as,

$$
a^{(N)\dagger}_{A_1} \dots S^{(N)\dagger}_{A_\Lambda} |0\rangle = \lambda_{\mathcal{F},N} |0\rangle \tag{69}
$$

with associated eigenvalue

$$
\lambda_{\mathcal{F},N} = \omega_{A_1,N} + \dots + \omega_{A_\Lambda,N}, \tag{70}
$$

corresponding to the set

$$\mathcal{F} = \{A_1, \dots, A_\Lambda\}. \tag{71}$$

This provides an indexing for the spectrum of $H^{\mathrm{B}}_{\mathrm{sc},N}$ in terms of finite subsets of $\mathbb{N}\times\mathbb{N}$ with at most Λ elements. Similarly, for the case of the limiting $H^{\mathrm{B}}_{\mathrm{sc}}$, the eigenfunctions are constructed in terms of creation operators, but now the sequences can be of arbitrary length. Thus, the eigenvalues are in one to one correspondence now with all finite subsets of $\mathbb{N}\times\mathbb{N}$.

For any given finite subset $\mathcal{F}$ of $\mathbb{N}\times\mathbb{N}$, we just have to choose N larger than the number of elements of $\mathcal{F}$ in order to ensure that $\mathcal{F}$ is also included in the indexing for the eigenvalues of $H^{B}_{\mathrm{sc},N}$. As $\mathcal{F}$ is finite, $\omega_{A,N} \to \omega_A$ and the expressions for the eigenvalues are finite sums,

$$\lambda_{\mathcal{F},N} \to \lambda_{\mathcal{F}}, \tag{72}$$

as required.

At this stage, we should make a remark on the multiplicity of the spectrum of H^{B}_{sc}. For a given index $A=(m,n)$, see (32),

$$\omega_A \geq \pi^2 \min\{R^1 l^1, R^2 l^2\}, \tag{73}$$

where the constants on the right hand side are independent of A. Then, for a given finite subset $\mathcal{F}$ with Φ elements,

$$\lambda_{\mathcal{F}} \geq \Phi\pi^2 \min\{R^1 l^1, R^2 l^2\}. \tag{74}$$

Hence, the class of subsets $\tilde{\mathcal{F}}$ such that $\lambda_{\mathcal{F}} = \lambda_{\mathcal{F}}$, is limited by the fact that $\tilde{\mathcal{F}}$ can not have more than $\lambda_{\mathcal{F}}/\pi^2 \min\{R^1 l^1, R^2 l^2\}$ elements. This ensures that each eigenvalue of H^{B}_{sc} is of finite multiplicity.

The above lemma shows that the spectra of $H^{\mathrm{B}}_{\mathrm{sc},N}$ converge to the spectrum of $H^{\mathrm{B}}_{\mathrm{sc}}$. However it does not provide information about the precise sense in which $H^{B}_{\mathrm{sc},N} \to H^{B}_{\mathrm{sc}}$ if at all the case. We may consider, for instance, computing the large N limit of the expectations of the solutions to the heat equation. The Hamiltonian H^{B}_{sc} is unitarily equivalent to a self-adjoint operator acting on the Hilbert space $L^2(\ell_2, \mathrm{d}\gamma)$, where $\mathrm{d}\gamma$ is a Gaussian measure on ℓ_2. Recall that ℓ_2 comprises square summable sequences $(Y^A)_1^\infty$ such that $\|Y\|^2 = \sum_A |Y^A|^2 < \infty$. A procedure for constructing Gaussian measures in ℓ_2 is described in the monograph [34].

For each wave function $\psi \in L^2(\ell_2, \mathrm{d}\gamma)$, there exists $\psi_N \in L^2(\mathbb{R}^\Lambda, \mathrm{d}\gamma^\Lambda)$ such that $\psi_N \to \psi$. Here $\mathrm{d}\gamma^\Lambda = e^{-\|Y\|^2/2}\mathrm{d}Y$ is the standard Gaussian measure in $\mathbb{R}^\Lambda$. By performing a suitable change of coordinates, the Hamiltonian $H^{\mathrm{B}}_{\mathrm{sc},N}$ is also an operator acting on $L^2(\mathbb{R}^\Lambda, \mathrm{d}\gamma^\Lambda)$. Hence, we are in the position of being able to compare the exact model with the regularized one. Indeed, since $L^2(\mathbb{R}^\Lambda, \mathrm{d}\gamma^\Lambda)$ are subspaces of $L^2(\ell_2, \mathrm{d}\gamma)$ via the natural identification

$$\phi(Y^A) \longmapsto \phi((Y^A), 0, \ldots), \qquad (Y^A) \in \mathbb{R}^\Lambda, \tag{75}$$

both operators $H^{\mathrm{B}}_{\mathrm{sc},N}$ and $H^{\mathrm{B}}_{\mathrm{sc}}$ act in the *same* subspace. Note that here we *must* use the fact that $\mathrm{d}\gamma$ is Gaussian in order to ensure that the right hand side is a member of the latter space. In the other direction, we have the projected states

$$\phi(Y^{(1,0)}, \ldots) \longmapsto \phi((Y^A), 0, \ldots) =: \phi_N(Y^{(1,0)}, \ldots). \tag{76}$$

for all $\phi(Y^{(1,0)}, \ldots) \in L^2(\ell_2, \mathrm{d}\gamma)$. This identification gives a precise meaning to the limit

$$\lim_{N\to\infty} \langle \psi, (H^{\mathrm{B}}_{\mathrm{sc},N}\phi_N - H^{\mathrm{B}}_{\mathrm{sc}}\phi)\rangle, \tag{77}$$

making it possible to verify rigorously whether $H^{\mathrm{B}}_{\mathrm{sc},N} \to H^{\mathrm{B}}_{\mathrm{sc}}$ in the weak topology.

For given initial states $\phi, \psi \in L^2(\ell_2, \mathrm{d}\gamma)$ and $t > 0$, we can also compute the limit

$$\lim_{N\to\infty} \langle \phi_N, e^{-H^{\mathrm{B}}_{\mathrm{sc},N} t} \psi_N \rangle \tag{78}$$

via the Mehler formula. For this we should recall that the heat kernel of the regularized semi-classical Hamiltonian can be found explicitly, in the literature $e^{-H^{\mathrm{B}}_{\mathrm{sc},N} t}$ is the famous Ornstein Uhlenbeck semi-group. This rises the question of whether the exact semi-group $e^{-H^{\mathrm{B}}_{\mathrm{sc}} t}$ could possibly be characterized using the Feynman-Kac formula.

Note that the characterization of the regularized Hamiltonian in the space with Gaussian measure is far more advantageous than our previous approach of using the space with Lebesgue measure. Indeed one can easily prove that it is not possible to construct a Lebesgue measure in ℓ_2.

6. Center of the group, mass gap and Confinement

Once all the previous spectral properties have been established, we would like to study the behavior of the supermembrane with central charges, or equivalently, the behavior of the symplectic NCSYM theory at low and high energies. It was an original idea of G. 't Hooft, [35, 36] that permanent quark confinement occurs in a gauge theory if its vacuum condensates into a state which resembles a superconductor. His proposal was to consider the confinement of quarks as dual of the Meissner effect, where the role of magnetism and electricity are interchanged. In his approach he considered a nonabelian gauge theory in terms of an abelian theory enriched with Dirac magnetic monopoles.

This is exactly what happens here, as we will see along the section, although the Yang-Mills theory that describes it is a symplectic non-commutative one. We are going to study the symmetries of the theory and by them, we will be able to identify the center of the group of the residual symmetry. We are going to show how it plays a role to create confinement and through its breaking how the theory enters in a quark-gluon plasma phase which corresponds to the supermembrane without central charges.

6.1. **Symmetries.** The $D = 11$ supermembrane in the light cone gauge with a Minkowski target space posses a residual invariance associated to the infinite group of area preserving diffeomorphisms $\mathbb{D}iff^{\infty}(\Sigma)$ on a Riemann surface Σ of genus g [3]. The supermembrane in eleven dimensions realizes through its Hamiltonian a subgroup of the full group of area preserving diffeomorphisms, which is the one associated to the diffeomorphisms connected to the identity, $\mathbb{D}iff^{\infty}_{\mathbb{I}}(\Sigma)$. They are associated to the exact 1-forms of the theory, [3]:

(79)
$$\partial_r(\sqrt{\omega(\sigma)}\xi^{rs}(\sigma,\tau)) \equiv D_r\xi^r(\sigma) = 0; \quad \rightarrow \quad \xi^r(\sigma) = \frac{\epsilon^{rs}}{\sqrt{\omega(\sigma)}}\partial_s\xi(\sigma)$$

whose composition law can be expressed in terms of Poisson brackets,

(80)
$$\xi_3 = \{\xi_2, \xi_1\}$$

Poisson brackets satisfy the Jacobi identity and a matrix regularization in terms of $SU(N)$ brackets can be performed. However when the supermembrane target space has a compactified sector $M_4 \times X_7$, think for simplicity in $M_4 \times S^1 \times \cdots \times S^1$, then the diffeomorphisms disconnected to the identity are realized in terms of harmonic one-forms over the Riemann surface. These are,

(81)
$$\Delta\widehat{X}^r(\sigma) = 0$$

closed but non-exact forms associated to the winding of the supermembrane. It was shown in [9] that the harmonic forms are realized at the level of the Hamiltonian description. The extra structure constants associated to two harmonic forms g^C_{rs} and to the mixing between the harmonic forms and the exact forms g^C_{rA} did not admit a $SU(N)$ regularization in general terms. However if we consider a topological condition on configuration space as in [12] a consistent regularization may be performed.

The presence of r closed but non- exact forms can be seen in the dual picture, that is, on 10D IIA description, as the existence of r $U(1)$ gauge fields due to the compactification. This means that the compactified supermembrane has the following gauge symmetries from the type IIA point of view,

$$Diff_{\mathbb{I}}^{\infty} \times U(1)^r \tag{82}$$

It happens that the gauge fields satisfy certain additional symmetry, associated to the harmonic forms. The hamiltonian has an additional invariance related to the symplectomorphism group $Sp(2g, \mathbb{Z})$. In the particular case when the compactified sector of target space is a 2-torus the symmetry is $Sp(2, \mathbb{Z}) \simeq SL(2, \mathbb{Z})$, the same symmetry that appears compactifying IIA/S^1 or equivalently IIB/S^1.

One way to realize this symmetry in our formalism as was pointed out in [16] and in Section 2, is to observe that, dX^r must satisfy condition (5)

$$\oint_{C_s} dX^r = S_s^r \tag{83}$$

where C_s is a basis of homology defined on the 2-torus. The matrix S_s^r satisfy that

$$S_r^t \epsilon^{rs} S_s^u = \epsilon^{tu} \tag{84}$$

that is $S_r^s \in SL(2, \mathbb{Z})$.

In the supermembrane with fixed central charges there are area preserving diffeomorphisms not homotopic to the identity corresponding to biholomorphic maps f, mapping Teichmuller space onto itself. It induces a map on the fundamental group Π_1 where the basis of homotopy is mapped by an element of $\frac{SL(2,\mathcal{C})}{Z_2}$. Under these conformal maps the basis of harmonic one-forms transform by an element of $\frac{SL(2,\mathcal{C})}{Z_2}$. They are area preserving diffeomorphisms for our choice of W, as was discussed in Section 2.

Although A, see eqn (7) is univalued over Σ it has an infinitesimal gauge transformation law that represents an unusual realization of the diffeomorphisms algebra,

$$A \to A + D\xi + \{A, \xi\} = A + \mathcal{D}\xi. \tag{85}$$

This transformation is generated by a first class constraint at exact and regularized levels. See Appendix. It corresponds to a symplectic connection preserving the symplectic structure of the fibers under holonomies. With this transformation the general structure of the first class constraint which generate the gauge symmetry of the theory close an algebra at the exact and also at the $SU(N)$ *regularized* model. Let us remark that the transverse modes transforms as usual,

$$\delta X = \{\xi, X\} \tag{86}$$

In order to construct the noncommutative gauge theory one has to fix the harmonic sector and the resulting symmetry is the center of $Sp(2, Z)$, which is $Z(2)$.

6.2. **The center as a mechanism for confinement at the exact level of the theory.** The way in which the central charge or its associated residual $Z(2)$ symmetry of the hamiltonian provide mass to the supermembrane may be described in terms of the quadratic derivative terms of the configuration fields X^m and A_r. The derivative terms correspond to the mapping of the target space to Σ induced by the minimal inmersion realized by $\widehat{X}_r$, $r = 1, 2$, the harmonic fields over Σ:

$$D_r Y_A = \{\widehat{X}_r, Y_A\} = \lambda_{rA}^B Y_B = \lambda_{rA} Y_A \tag{87}$$

where

$$\lambda_{rA}^B = \int d^2\sigma \sqrt{\omega}\{\widehat{X}_r, Y_A\} Y^B \tag{88}$$

and corresponds to a particular subset of the structure constant that mix harmonic and exact forms g_{rA}^C. For the case of a torus a explicit relation were found in [12]. The quadratic terms on the derivatives of the configuration variables define a strictly positive potential whose contribution to the overall potential gives rise to a basin shaped potentail, eliminating the string-like spikes and providing a discrete spectrum even for the supersymmetric model.

Without the central charge, on the directions where the quartic potential vanishes the SUSY contribution to the potential renders an unbounded from below potential and a continuous spectrum. The quantum mass is bounded by below by its semi-classical contribution as has already been shown in the preceding sections, then this means that the center created by a discrete symmetry once that a topological condition is implemented in the model, is a mechanism for giving mass to the monopoles.

We ask ourselves what happens when we enlarge the topological condition due to compactification into T^6 for example. The size of the symplectic group increases, however as explained in [38] the size of the center of the group remains constant $Z(2)$ in distinction to $SU(N)$ gauge groups. In there from a lattice point of view $Sp(2), Sp(3)$ were used to show de-confinement transition phase by the breaking of the center.

6.3. **Center of the group as a mechanism for confinement in the $SU(N)$ formalism.** The center of the group in $SU(N)$ regularization is known to be $\mathbb{Z}_N$. In this case since the origin is a inherited structure of topological origin created by the monopoles induced in the torus the real discrete symmetry is $\mathbb{Z}_N \times \mathbb{Z}_N$, where an element belonging to the center satisfies that

$$\{z \in \widehat{\mathbb{Z}}_N \times \widehat{\mathbb{Z}}_N | \quad \widehat{z}^N = 1; z = e^{\frac{2\pi i (V_r \wedge A)}{N}}\} \tag{89}$$

The terms associated to the mass terms are defined in terms of a regularized object found in ([12])which correspond to an specific choice of the structure constant parameters. In terms of the $SU(N)$ basis is :

$$\widehat{\lambda}_{rA} \equiv Tr([T_{V_r}, T_A]T_{-V_r-A}) = 2iN\sin(\frac{V_r \wedge A}{N}\pi) \tag{90}$$

where T_{V_r} correspond to two particular matrices of T_A, in which the $SU(N)$ algebra can be expanded. In this terms the center corresponds to

$$\widehat{z}_r \equiv \frac{1}{N^4} Tr(T_{V_r} T_A T_{-V_{r-A}}) \tag{91}$$

Following [12], they correspond specifically to $T_{V_1} = T_{0,1}, T_{V_2} = T_{1,0}$, were we have used for the definition of $T_A = N z^{1/2 a_1 a_2} P^{a_1} Q^{a_2}$ as in ([35],[4]). The lambda contribution then can be easily re-expressed as,

$$\widehat{\lambda}_{rA} = Im(\widehat{z}_r). \tag{92}$$

The generation of the discrete mass spectrum is analogous to what happens for the exact theory.

The eigenvalues of the Hamiltonian are bounded from below by those of the semi-classical spectrum in such a way that the mass terms are created by the center whose unitary realization on the Hilbert space of states commutes with the hamiltonian and it represents then an unbroken symmetry.

6.4. **Confinement, screening and phase transition.** As we have already seen the mass terms are determined by the elements of the center $m(z)$ associated to $z = Tr(T_{V_r} T_A T_{-V_r-A}) \in \mathbb{Z}$. Since T_{V_r} appears in the regularized model as a consequence of $\widehat{X}_r$ which are the harmonic forms associated to the winding and defining the monopole charge $\{\widehat{X}_r, \widehat{X}_s\} = \epsilon_{rs} n$, then if the monopole charge disappears the center becomes $z = T_A T_{-A} = T_{(0,0)} = 1$ trivial. This is what we expect for the de-confined phase a breaking of the center of the group. This effect in the same way can also be seen at the level of the exact theory. Then we have two pictures, one in which the correlation length of the particles is the inverse of the mass of the glueball states $\xi_C = 1/m_{eff}$, and we can define an effective volume $V_{eff} = R/\xi_C$. There the particles feel the topological effects and get confined. It corresponds to the supermembrane with central charges. Other regime in which $m^2 = 0$, $\xi \to \infty$ and $V_{eff} = 0$ in which the particles loss the information that they are confined in a boundary with topological condition and behave as in a quark-gluon plasma. In the supersymmetric picture, the Hamiltonian corresponding to the $N = 1$ supermembrane with central charges,

$$H_1 = \Delta_1 + V_1^B + V_1^F \tag{93}$$

has purely discrete spectrum at quantum level due to the presence of a nontrivial central charge in the algebra of supersymmetry. It admits an interpretation as a first quantized theory, however the Hamiltonian corresponding to the $N = 4$ compactified supermembrane without central charges,

$$H_2 = \Delta_2 + V_2^B + V_2^F \tag{94}$$

has a continuum spectrum. The supermembrane is interpreted as a many-body object which fluctuates into different vacua were the number of particles nor the topology of the membrane is not conserved [2, 10]. We conjecture that it describes the quark-gluon plasma. Moreover, along the flat directions, corresponding to the commutative picture, the particles can behave as free particles since the potential vanishes. The particles *donot* feel any force between them. This is what we expect in the asymptotic free regime of a susy QCD. Then the transition happens due to a quantum change in the irreducible winding, that is although both membranes are compact and can have the same topology , i.e. a torus, there is a change in the topological condition of quantum nature. It corresponds to the sequence

$$U(1) \times U(1) \to U(1) \to Z(2) \tag{95}$$

which corresponds to a monopole bounding two strings. We may well conjecture about the origin of this phenomenon of quantum nature. This effect happens because lowering the energy scale becomes more advantageous for the membrane to have an irreducible wrapping. This effect may well happen because since the radius of the compactified extra dimension becomes smaller as we lower the scale, there is a critical scale at which the area of the supermembrane is minimized not by wrapping in a cycle but doing on a calibrated submanifold generated by the monopoles dual to the irreducible wrapping. From the quantum topology change in (2+1)d see [45, 43].

6.5. **Supermembrane origin and interpretation of susy QCD.** We would like to stress that in our picture, confinement is due mainly to two different facts: one is supersymmetry, and the other is extra dimensions. Due to the topological condition on the extra dimensions which we conjecture it naturally appears when the size of the extra dimensions become smaller, at a critical energy, the supersymmetry is broken. In fact, the topological condition correponds to the presence of a central charge of the supersymmetric algebra and gives mass to the gluons that enter in a confined phase. Since the magnetic flux is confined on the monopoles picture (as originally explained in [36]) the electric flux between them gets also confined forming a Z_2-string at the ends of which quarks get attached [48]. When one tries to separate, one needs to provide a force that is proportional to the force needed for increasing the effective radius of the compact dimensions, which grows linearly with the radius. That is, the confinement of quarks is due to the fact that extra dimensions are compactified and the supermembrane has an irreducible wrapping around them. To separate them implies to decompactify the space. In higher energies the size of the effective radius of the extra dimensions becomes bigger and the supermembrane does not get minimized its energy with and irreducible wrapping around them, (which corresponds to wrap a calibrated submanifold), but just they wrap cycle that minimize their volume. It is known that the presence of topological defects can diminish the energy of the vacuum and this is what happens in our case. Without them just through ordinary compactifications it allows to have degenerate points on its metric. Changes in the metric and topology are also allowed in classical analysis of GR [46], and has being studied in several papers [45, 43, 44]. That is, a change in its topology happens [45], the monopole picture is lost, the center becomes trivial as we have seen above, and the theory enters in the phase of asymptotic freedom in which the supermembrane can not be associated to a single particle but a many body object, as originally pointed by [2], which has continuous spectrum and the quarks feel free. This corresponds to see inside the hadron, that is shorter scale. The quarks-gluons form a plasma that does not feel the boundary effects since the correlation length becomes infinite and the effective volume is zero. We would like to point out another natural explanation that emerges from here: supersymmetry is the intrinsic origin of the topological condition. We conjecture that maybe this is the natural way in which supersymmetry breaking is realized in the nature. In the way we make the compactification we do not obtain exotic matter as in KK reduction but the effect is to give mass (without a Higss mechanism) to the scalar fields at the same time that we break supersymmetry. So we speculate that supersymmetry, membrane description and extra dimensions would be the hidden reason for QCD behavior in both phases: the confined one and the asymptotic freedom.

7. Discussion and Conclusions

We obtained a bound in the operator sense for the bosonic Hamiltonian of the $D = 11$ Supermembrane with central charges. The Hamiltonian is bounded from below by a strictly positive constant times the Hamiltonian of an harmonic oscillator. The bound implies that the resolvent of the Hamiltonian is a compact operator. In particular it implies that its spectrum is discrete with finite multiplicity and it contains a mass gap. It is the first result in the literature concerning the spectrum of the supermembrane theory, all previous results describe properties of $SU(N)$ regularizations of the theory. The proof extends to the exact infinite dimensional theory a similar bound we already obtained for the $SU(N)$ regularized model. In that case the bound was used to prove that the fermionic potential does not change the qualitative quantum properties of the bosonic Hamiltonian. The heat kernel of the regularized Supermembrane with central charges was rigorously obtained, convergence in terms Schatten-Neumann norms was proven implying a well defined Feynman formula for the heat kernel. The large N limit of that formula is expected to converge to the Feynman integral of the supermembrane with central charges. Since this theory is quantum equivalent to a symplectic noncommutative SYM theory on $(2 + 1)$ dimensions, with a compact without boundary space-like manifold, the same properties are valid for these theories. The $N = 1$ symplectic Yang-Mills in $2+1$D is coupled to some scalar fields coming from the dimensional reduction of NCYM theory in $10D$. We recall that the degrees of freedom of both theories are the same. We consider that this as a one step forward the quantization of M-theory.

We show that the supermembrane theory when compactified in $4D$ can be interpreted as a theory modeling susy QCD. It exhibits confinement in the phase at zero temperature since the theory becomes naturally the $N = 1$ supermembrane with central charges. By rising the energy the theory enter in the phase of asymptotic freedom described by the $N = 4$ compactified supermembrane without central charges. The phase transition is described by the breaking of the center of the group as we have explicitly showed in the previous section. We conjecture a possible reason why this phase transition can happen:

At high energies the size of the effective radius of the extra dimensions become bigger and the irreducible wrapping of the supermembrane on a calibrated submanifold of the target becomes a reducible wrapping on the compact sector of the target space with zero central charge. This corresponds to see inside the hadron, that is shorter scale. The quarks-gluons form a plasma that does not feel the topological effects since the correlation length becomes infinite and the effective volume is zero. Along the commutative directions the quarks experiment no force.

8. Apendix

We are going to give an explicit calculation of the $SU(N)$ gauge symmetry in the regularization of the $D = 11$ supermembrane with central charges. The general structure of the first class constraints which generate the gauge symmetry of the regularized model arising from the $D = 11$ supermembrane with central charges is,

$$\phi^D \equiv \lambda f^D_{E-w_s,w_s}\Pi^{Es} + f^D_{A+w_s,F-w_s}A^A_s\Pi^{Fs} = 0 \tag{96}$$

where λ is an arbitrary constant parameter and f^C_{AB} are the $SU(N)$ structure constants. The algebra associated to the first class constraints is obtained by considering the Poisson brackets of the generators. We have,

(97)

$$[\lambda f^D_{E-w_s}, w_s\Pi^{Es} + f^D_{A+w_s,F-w_s}A^A_s\Pi^{Es}, f^C_{L-w_r}, w_r\Pi^{Lr} + f^C_{F+w_r,E-w_r}A^F_r\Pi^{Er}]_P =$$
$$\lambda f^D_{E-w_s,w_s}f^C_{-E+w_s,F-w_s}\Pi^{Fs} - \lambda f^D_{-L+w_s,F-w_s}f^C_{L-w_r,w_r}\Pi^{Fs}$$
$$+ f^D_{A+w_s,E-w_s}f^C_{-E+w_s,F-w_s}A^A_s\Pi^{Fs} - f^D_{E+w_s,F-w_s}f^C_{A+w_s,-E-w_s}A^A_s\Pi^{Fs}.$$

We now use the explicit expression for the structure constants, for the terms depending on λ we get after some calculations,

$$\lambda\delta^{D+C}_F\Pi^{Fs}N^2\sin(\frac{(C\wedge D)}{N}\pi)\sin(\frac{(D+C)\wedge w_s}{N}\pi) \tag{98}$$

which may be re-written as

$$\lambda f^C_{-D,E}f^E_{F-w_s,w_s}\Pi^{Fs} \tag{99}$$

while for the remaining terms we get

$$\begin{aligned} N^2\delta^{C+D}_{A+F}&(\sin(\frac{(A+w_s)\wedge D}{N}\pi))(\sin(\frac{C\wedge(F-w_s)}{N}\pi)) \\ &-(\sin(\frac{D\wedge(F-w_s)}{N}\pi))(\sin(\frac{(A+w_s)\wedge C)}{N}\pi)) \end{aligned} \tag{100}$$

which may be re-written as

$$f^C_{-D,E}f^E_{A+w_s,F-w_s}A^A_s\Pi^{Fs} \tag{101}$$

we then get, for any λ and any N,

$$[\phi^D, \phi^C]_{Poisson} = f^C_{-D,E}\phi^E \tag{102}$$

where as before f are the $SU(N)$ structure constants.

The same algebra is valid when we take the large N limit. In fact, if we take the same constraints but with the structure constants of the area preserving diffeomorphisms instead of the $SU(N)$ ones, we obtain the corresponding algebra (102). In the $N \to \infty$ limit there is an equivalent realization of the generators in terms of the constraints,

$$\widetilde{\phi}^D = \lambda f^D_{E-w_s,w_s}\Pi^{Es} + f^D_{A,F}A^A_s\Pi^{Fs} = 0 \tag{103}$$

they also satisfy;

$$[\widetilde{\phi}^D, \widetilde{\phi}^C]_{Poisson} = f^C_{-D,E}\widetilde{\phi}^E. \tag{104}$$

In particular, λ may be taken to be 1. In the regularized model in [12] we took the $N = \infty$ model in terms of the decomposition on an orthonormal basis over the Riemann surface, we fixed the gauge ad then obtained a regularized model. An interesting alternative approach could be to obtain the regularized model satisfying the symmetry generated by ϕ^D and then to perform the gauge fixing.

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